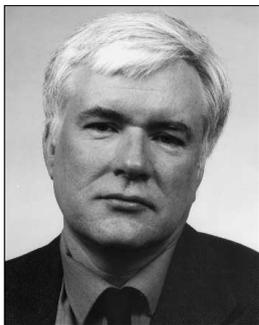


Teaching Symmetry in the Introductory Physics Curriculum

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Symmetry is a crucial concept in mathematics, chemistry, and biology. Its definition is also applicable to art, music, architecture, and the innumerable patterns designed by nature, in both animate and inanimate forms. In modern physics, however, symmetry may be the most crucial concept of all. Fundamental symmetry principles dictate the basic laws of physics, control structure of matter, and define the fundamental forces in nature.

Some famous mathematicians and physicists comment on symmetry:

"I aim at two things: On the one hand to clarify, step by step, the philosophic-mathematical significance of the idea of symmetry and, on the other, to display the great variety of applications of symmetry in the arts, in inorganic and organic nature."¹ — *Hermann Weyl*

"Special relativity emphasizes, in fact is built on, Lorentz symmetry or Lorentz invariance, which is one of the most crucial concepts in 20th Century Physics."² — *C. N. Yang*
(*Nobel Laureate in Physics*)

"Symmetry is fascinating to the human mind; everyone likes objects or patterns that are in some way symmetrical ... but we are most interested in the symmetries that exist in the basic laws themselves."³

— *Richard P. Feynman* (*Nobel Laureate in Physics*)

"I heave the basketball; I know it sails in a parabola, exhibiting perfect symmetry, which is interrupted by the basket. It's funny, but it is always interrupted by the basket."⁴

— *Michael Jordan*
(*retired Chicago Bull*)

Today we understand that all of the fundamental forces in nature are unified under one elegant symmetry principle. We revere the fundamental symmetries of nature and we have come to intimately appreciate their subtle consequences. As we will see, to succumb to a crackpot's invention requiring us to give up the law of energy conservation would be to give up the notion of a symmetry principle, that time flows with no change in the laws of physics. Symmetry controls physics in a most profound way, and this was the ultimate lesson of the twentieth century.

Yet, even a sampling of the crucial role of symmetry in physics is completely omitted in teaching beginning students, not only from the high-school curriculum, but also from the traditional first-year calculus-based college physics course. It does not appear in the National Science Education Standards.

It is possible, nonetheless, to incorporate some of the underlying ideas of symmetry and its relationship to nature into the beginning courses in physics and mathematics at the high-school and early college level. The ideas are really not that difficult. When the elementary courses are spiced with these ideas, they begin to take on some of the dimensions of a humanities or fine-arts study: *Symmetry is one of the most beautiful concepts, and its expression in nature is perhaps the most stunning aspect of our physical world.*

What follows is a description of a high-school module that introduces the key ideas that, in many examples, tie physics to astrophysics, biology, and chemistry. It also reveals some of the modern thinking in a conversational way. We are experimenting with the implementation of this approach in the classroom, in Saturday Morning Physics at Fermilab, and elsewhere.

A lot more material that cannot be presented in this brief article can be found at our symmetry website, www.emmynoether.com. We continually update our website as our educational experiment in symmetry proceeds.

We encourage you, and your students, to visit it. And don't hesitate to send us suggestions, comments, and even kindly worded complaints.

What Is Symmetry?

When a group of students is asked to define "symmetry," the answers they give are generally all correct. For example, to the question "what is symmetry?" we hear some of the following:

- ▶ "it's like when the sides of an equilateral triangle are all the same, or when the angles are all the same..."
- ▶ "things are in the same proportion to each other..."
- ▶ "things that look the same when you see them from different points of view..."

From the many diverse ways of describing symmetry, we quickly reach agreement with the scientists' definition:

- ▶ "Symmetry is an invariance of an object or system to a set of changes (transformations)."

In simpler language, a thing (*a system*) is said to possess a symmetry if we can make a change (*a transformation*) in the system such that, after the change, the thing appears exactly the same (*is invariant*) as before. Let us consider some examples.

Translations in Space

A physical system can simply be moved from one place to another place in space. This is called a "spatial translation."

Consider a classroom pointer. Usually it is a wooden stick of a fixed length, about one meter. We can translate the pointer freely in space. Do its physical properties change as we perform this translation? Clearly they do not. The physical material, the atoms, the arrangement of atoms into molecules, into the fibrous material that is wood, etc., do not vary in any obvious way when we translate the pointer. This is a symmetry: it is a statement that the laws of physics themselves are symmetrical under translations of the system in space. Any *equation* we write describing the quarks, leptons, atoms, molecules, stresses and bulk

moduli, electrical resistance, etc., of our pointer must *itself* be invariant under translation in space.

For example, we can easily write a formula for the "length" of the pointer that is independent of where the pointer is located in space (we leave this as an exercise, or see our website). Such a formula contains the information that the length of the pointer, a physical measure of the pointer, doesn't change under translations in space. Or, put another way, the formula is "invariant under spatial translations." While this would be a simple example, the (highly nontrivial) assumption is that *all correct equations in physics are translationally invariant!* Thus, if we have a physics laboratory in which all kinds of experiments are carried out and all sorts of laws of nature are discovered and tested, the symmetry dictates that the same laws will be true if the laboratory itself is moved (*translated*) to another location in space.

The implications of this "oh so simply stated" symmetry are profound. It is a statement about the nature of space. If space had at very short distances the structure of, e.g., a crystal, then moving from a lattice site to a void would change the laws of nature within the crystal. The hypothesis that space is *translationally invariant* is equivalent to the statement that one point in space is equivalent to any other point; i.e., the symmetry is such that translations of any system or, equivalently, the translation of the coordinate system, does not change the laws of nature. We emphasize that *this is a statement about space itself*; one piece of space is as good as another! We say that space is smooth or homogeneous (Einstein called it a "continuum"). Equivalently, the laws and the equations that express these laws are invariant to translations, i.e., possess translational symmetry.

Now, we can get confused in applying translational invariance. Consider an experiment to study the translational symmetry of the electric charge by measuring, e.g., acceleration of electrons in a cathode ray tube. If there were a huge magnet outside the laboratory, then the experimental results would change when the tube is moved around inside the lab. This is not, however, a violation of translational symmetry, because we forgot to include the magnet in the move. If we live in a region of space with intrinsic magnetic fields, then we might detect dependence upon



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position, and the symmetry would not seem to exist. However, it is our belief that flat space is smooth and homogeneous. The most profound evidence comes later.

Translations in Time

The physical world is actually a fabric of events. To describe events we typically use a three-dimensional coordinate system for space, but we also need an extra one-dimensional coordinate system for time. This is achieved by building a clock. The time on the clock, together with the three-dimensional position of something, forms a four-coordinate thing (x, y, z, t) , called an “event.” (Note: we always assume that the clock is ideally located at the position of the event, so we don’t get confused about how long it takes for light to propagate from the face of a distant clock to the location of the event, etc.) Some examples of events: (i) We can say that there was the event of the firecracker explosion at (x_F, y_F, z_F, t_F) ; (ii) The N.Y. Yankees’ third baseman hits a fast pitch at (x_H, y_H, z_H, t_H) ; (iii) Neil Armstrong’s foot first touched the surface of the Moon at the event (x_M, y_M, z_M, t_M) .

Now we have the important symmetry hypothesis of physics: *The laws of physics, and thus all correct equations in physics, are invariant under translations in time.* That is, to all of our fabric of events, such as the events we described above, we can just shift every time coordinate by an overall common constant. Mathematically, we replace every time t_i for every physical event by a new value $t_i + T$. The T ’s cancel in all correct physics equations; the equations are all *time translationally invariant!* Time, we believe, is also smooth and homogeneous.

Indeed, the constancy of the basic parameters of physics, e.g., electric charge, electron mass, Planck’s constant, the speed of light, etc., over vast distances and times has been established in astronomical and geological observations to a precision approaching 10^{-8} over the entire age of the Universe.⁵ The laws of physics appear to be constant in time. The experimental evidence is very strong!

Rotations

A sphere (or a spherical system, or Michael Jordan’s basketball) can be rotated about any axis that passes through the center of the sphere. The rotation angle can be anything we want, so let’s take it to be 63° . After this rotation (often called an “operation” or “transformation”) the appearance of the sphere is not changed. We say that the sphere is “invariant” under the “transformation” of rotating it about the axis by 63° . Any mathematical description we use of the sphere will also be unchanged (invariant) under this rotation. There are an infinite number of symmetry operations that we can perform upon the sphere. Furthermore, there is no “smallest” nonzero rotation that we can perform; we can perform “infinitesimal” rotations of the sphere. We say that the symmetry of the sphere is “continuous.”



Fig. 1. Emmy Noether, which rhymes with “mother,” (1882-1935) practiced at Göttingen where mathematicians Hilbert and Klein and physicists Heisenberg and Schroedinger were professors. Fleeing the rise of Naziism, she spent her last few years in the United States at Bryn Mawr College and the Institute for Advanced Study at Princeton University.⁶ Photo courtesy of the Special Collections Department, Bryn Mawr College.

Consider again our classroom pointer. We can rotate the pointer freely in space. Do its physical properties change as we perform this rotation? Clearly they do not. This too is a symmetry: it is a statement that the laws of physics themselves are symmetrical under rotations in space. Under rotations in free space the length of our classroom pointer, R , doesn’t change.

We could actually perform a mathematical rotation about the origin of our coordinate system in which we have written a formula for the length of a pointer. We would find that the formula doesn’t change (just the coordinates, the things the formula acts upon, do; this isn’t hard to see, and we do it on the website). We say that the length of the pointer is invariant under rotations. Indeed, it is our firm belief that *the laws of physics, and thus all correct equations in physics, are invariant under rotations in space.* This is, again, based upon experimental data. It is a statement about the nature of space; space is said to be *isotropic*, that is, all directions of space are equivalent.

In summary: *The laws of physics are invariant under spatial and temporal translations, and rotations in space.*

Needless to say, there are many additional symmetries, some of which we will discuss later.

Symmetries of the Laws of Physics and Emmy Noether’s Theorem

In 1905, mathematician Emmy (Amalie) Noether, (Fig. 1), proved the following theorem:

For every continuous symmetry of the laws of physics, there must exist a conservation law.

For every conservation law, there must exist a continuous symmetry.

Thus, we have a deep and profound connection between a symmetry of the laws of physics and the existence of a corresponding conservation law. In presenting Noether’s theorem at this level, we usually state it without proof. (A fairly simple proof can be given if the student is familiar with the action principle; it can, however, be motivated with simple examples, as we do below.)

Conservation laws, including the conservation of energy, momentum, and angular momentum (these are the most famous), are studied in high school. They are usually presented as consequences of Newton's laws (which is true). We now see from Noether's theorem that they emerge from symmetry concepts far deeper than Newton's laws.

Now, as we have stated above, it is an experimental fact that the laws of physics are invariant under the symmetry of spatial translations. This is a strong statement. What is the physical consequence of this? Thus comes the amazing theorem of Emmy Noether, which states, in this case:

The conservation law corresponding to space translational symmetry is the Law of Conservation of Momentum.

So, we learn in senior physics class that the total momentum of an isolated system remains constant. The i th element of the system has a momentum in Newtonian physics of the form $\vec{p}_i = m\vec{v}_i$, and the total momentum is just the sum of all of the elements,

$$\vec{P}_{total} = \vec{p}_1 + \vec{p}_2 + \dots + \vec{p}_N \quad (1)$$

for a system of N elements. Noether's theorem states that \vec{P}_{total} is conserved; i.e., it does not change in time, no matter how the various particles interact, because the interactions are determined by laws that don't depend upon where the whole system is located in space!

Note that momentum is, and must be, a vector quantity (hence the little arrow, $\vec{}$, over the stuff in the equations). Why? Because momentum is associated with translations in space, and the directions you can translate (move) a physical system form a vector! So, if you remember the Noether theorem, you won't forget that momentum is a vector when taking an SAT test!

Turning it around, the validity of the law of conservation of momentum as an observational fact, via Noether's theorem, supports the hypothesis that space is homogeneous, i.e., possessing translational symmetry. The more we verify the law of conservation of momentum (and it has been tested literally trillions of times in laboratories all over the world, at all distance scales), the more we verify the idea that space is homogeneous, and not some kind of crystal lattice!

We have also stated above that the laws of physics are invariant under translations in time. What conservation law then follows by Noether's theorem? Surprise! It is nothing less than the law of conservation of energy:

The conservation law corresponding to time translational symmetry is the Law of Conservation of Energy.

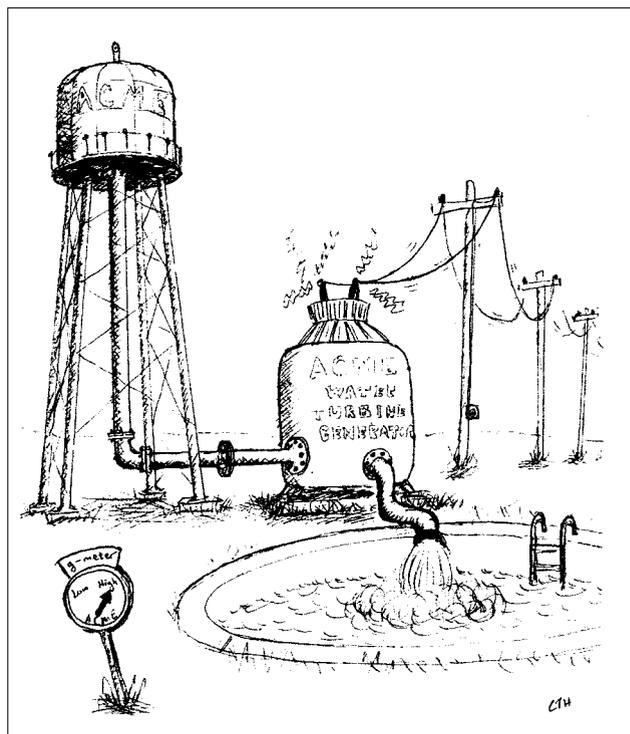


Fig. 2. Water is drained through turbine generator on days when the gravitational field strength is $g = g'$ and energy is produced and sold to power company. On days when gravitational field strength is $g < g'$ the water is pumped back up into the tower at a reduced cost in energy. Hence net energy is available from the system if g is time dependent.

Since the constancy of the total energy of a system is extremely well tested experimentally, this tells us that nature's laws are invariant under time translations.

Here is a cute example of how time invariance and energy conservation are interrelated. Consider a water tower that can hold a mass M of water and has a height of H meters. Assume that the gravitational constant, which determines the gravitational field strength, is g on every day of the week except Tuesday, when it is a smaller value $g' < g$. Now, we run water down from the water tower on Monday through a turbine (a fancy water wheel) generator that converts the potential energy MgH to electrical current to charge a large storage battery, (Fig. 2). We'll assume 100% efficiencies for everything, because we are physicists. This is Monday's job. For Tuesday's job we pump the water back up to H , using the battery power that we accumulated from Monday's job to run the pump. But now the g' value is smaller than g and the work done is $Mg'H$, which is now much less than the energy we got from Monday's job. This leaves us with $M(g - g')H$ extra energy still in the battery, which we can sell to a local power company to live on until next Monday. This is a perpetual motion machine! It produces energy for us, and we can convert that to cash. It does not conserve energy because we cooked up false laws of physics, in this case gravity, that are not time translationally invariant! Hence, we violated a precept of Noether's theorem. (Can you

come up with a similar cute example of violating momentum conservation by making the laws of physics spatially inhomogeneous?)

We also live in a world where the laws of physics are rotationally invariant:

The conservation law corresponding to rotational symmetry is the Law of Conservation of Angular Momentum.

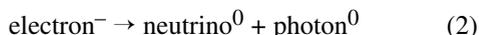
Conservation of angular momentum is often demonstrated in lecture by what is usually called “the three-dumbbell experiment.” The instructor stands on a rotating table, his hands outstretched, with a heavy dumbbell in each hand (who is the third dumbbell?), (Fig. 3). He turns slowly, and then brings his hands (and dumbbells) close to his body, (Fig. 4). His rotation speed (angular velocity) speeds up substantially. What is kept constant is the angular momentum, J , the product of I , the moment of inertia, times the angular velocity ω . By bringing his dumbbells in close to his body, I is decreased. But J , the angular momentum, must be conserved, so ω must increase. Skaters do this trick all the time.

Atoms, elementary particles, etc., all have angular momentum. The intrinsic angular momentum of an elementary particle is called spin. In any reaction or collision, the final angular momentum must be equal to the initial angular momentum. Like our planet Earth, particles spin and execute orbits and both motions have associated angular momentum. Data over the past 70 or so years confirm conservation of this quantity on the macroscopic scale of people and their machines and on the microscopic scale of particles. And now (thanks to Emmy, one of the greatest mathematicians of the century), we learn that these data imply that space is isotropic: all directions in space are equivalent.

The translational and rotational symmetries of space and time need not have existed. But that’s the way nature is. These are some of the actual properties of the basic concepts we use to describe the world: space and time.

Beyond

We have described how the fundamental conservation laws of everyday physics follow from the continuous symmetry properties of space and time. There are, however, many other conservation laws that are not usually studied in a first-year physics course. A simple example is the conservation of electric charge in all reactions. The total electric charge in an isolated system is a constant in time. For example, processes like:



(where superscripts denote charges), in which case electric charge could completely disappear, are forbidden. On

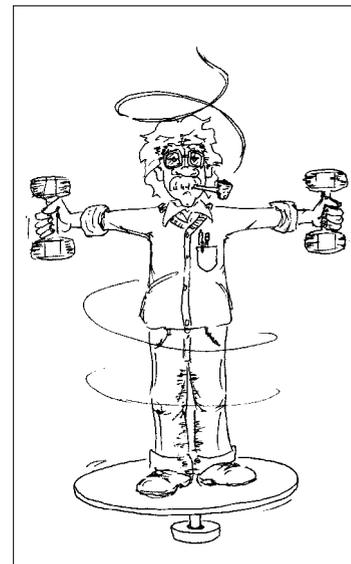
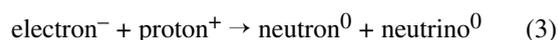


Fig. 3. The Professor with dumbbells rotates slowly when his arms are outstretched.



Fig. 4. Pulling the dumbbells close to his body reduces the moment of inertia, but angular momentum is conserved; hence the Professor rotates faster.

the other hand, processes like this one do occur:



Since the final state is electrically neutral, the negative electric charge of the electron must be identically equal and opposite to that of the proton to an infinite number of significant figures. Indeed, we can place a large quantity of hydrogen gas into a container and observe to a very high precision that hydrogen atoms (which are just bound states of $e^- + p^+$) are electrically neutral.

This conservation law, by Noether’s theorem, also arises from a profound symmetry of nature called *gauge* symmetry. This is an example of an abstract symmetry that does not involve space and time. In the late twentieth century we have come to realize that all of the forces in nature are controlled by such gauge symmetries. Gauge symmetry is very special, and it actually leads us to the complete theory of electrons and photons, known as (quantum)

electrodynamics, which has been tested to 10^{-12} precision. This is the most accurate and precise theory of nature that humans have ever constructed. Perhaps the most stunning result of the past century has been the understanding that all known forces in nature are described by gauge symmetries.

Einstein's special theory of relativity is all about relative motion and is based upon a fundamental symmetry principle about motion itself. This is a statement that the laws of physics must be the same for all observers, independent of their state of uniform motion. This symmetry principle of relativity can be expressed in a way that shows that it is a generalization of the concept of a rotation. The time interval between two events that occur at the same point in space is called the "proper time." The proper time can be expressed, like the length of our pointer, in such a way that it is invariant under motion.⁷ A formula can be written that relates the coordinate systems of two observers moving relative to one another, in terms of their relative velocity v . The formula is called a Lorentz transformation and it mixes time and space, much like a rotation in the xy plane mixes x and y . Like a rotation, it leaves the proper time invariant. Thus motion can be thought of as a rotation in space and time! Unfortunately, we must send you off to a textbook on special relativity (or our website) to learn about all of the miraculous effects that occur as a consequence of this. Relativity is an expansion of our understanding of the deep and profound symmetries of nature.

Other extremely important symmetries arise at the quantum level. A simple example is the replacement of one atom, say a hydrogen atom sitting in a molecule, by another identical atom. This is a symmetry because all hydrogen atoms are exactly the same, or *identical* in all respects. There are no warts or moles or identifying body markings on hydrogen atoms, or any other atomic scale particle for that matter, such as electrons, protons, quarks, etc. The symmetry associated with exchanging positions or motions of identical atoms or electrons or quarks has profound effects upon the structure of matter, from the internal structure of a nucleus of an atom, to the properties of everyday materials like metals, insulators, and semiconductors, to the external structure of a neutron star or white dwarf star. This identical-particle symmetry explains nothing less than the Periodic Table of the Elements; i.e., how the motion and distributions of the electrons are organized within the atoms as we go from hydrogen to uranium! All of chemistry is controlled by the interactions of electromagnetism together with the symmetry of identical particles.

In a one-week module we can develop these and other important symmetries, such as mirror symmetry, time

reversal symmetry, and the symmetry between matter and antimatter (we can even explain why antimatter exists, from the symmetry of relativity and Noether's theorem!). Conservation laws that are associated with more abstract symmetries, such as "quark color" and "supersymmetry," can also be illustrated and discussed. Indeed, this leads us to the frontier of theoretical physics; e.g., superstrings, M-theory, and the deeply disturbing open questions such as "(why) is the Cosmological Constant zero?" As we progressively proceed to the deepest foundations of the structure of matter, energy, space, and time, we must be more descriptive for our beginning students, but we become more thrilled, enchanted and excited by the fundamental symmetries that control the structure and evolution of our Universe.

Let this provocative conclusion close this very incomplete survey of the role of symmetry in physics. Please visit www.emmynoether.com for a much expanded version of this brief article.

Acknowledgment

We wish to thank Shea Ferrel for Figs. 3 and 4.

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